Graduate Seminar Applied Logic (S4A6) WiSe 2026–27

Model Theory of Valued Fields

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Time and Place. Friday 10–12, SemR 0.007

Organisation. There will be an organisational meeting via Zoom on Monday, 20 July 2026, 10 (c.t.), `https://uni-ms.zoom-x.de/j/63181643745`, *Meeting ID: 631 8164 3745*

In case you would like to give a talk, please send an email to `ludwigsm160@gmail.com` by Friday, 14 August 2026, indicating which talks you are interested in. If possible, please list at least three topics. If you have any questions, please feel free to contact us as well.

Abstract. This seminar gives an introduction to the model theory of valued fields. Valued fields play a central role in modern model theory. The first-order behaviour of a valued field is often controlled by two simpler pieces of data: its residue field and its value group. This idea is made precise in Ax–Kochen–Ershov type principles and in quantifier elimination theorems such as those of Robinson, Macintyre, and Pas. The seminar gives an introduction to these themes. We begin with the basic algebraic theory of valued fields and ordered abelian groups, then discuss algebraically closed valued fields as well as the famous AKE principle. We then turn to the model theory of $\mathbb{Q}_p$, definability questions over p-adic fields, and transfer results for NIP. Depending on the interests of the participants, the final part of the seminar may also cover metric valued fields in continuous logic.

Rules.

(1) Approximately one week before your talk, we will meet to discuss the content and structure of the presentation. Please prepare a first draft of your talk by then. Feel free to contact me earlier if mathematical or organisational questions arise.

(2) In addition to giving a 90-minute talk, each participant who wants to receive credit is required to submit a written summary of at least four pages on their topic by the time of their talk.

(3) Your grade will be determined, among other things, by the mathematical content of your talk, your understanding of the material, the pedagogical effort, the quality of the presentation, and the written summary. Your preparation for the meeting before the talk will also be taken into account.

Prerequisites. This seminar is designed for students who have already taken a first course in model theory, for example Mathematical Logic (V3A5/F4A1–5) during the Winter Semester 2025/26. In addition, only knowledge from a basic algebra course (covering e.g. field extension, ideals,...) is assumed.

Participants are however highly encouraged to make themselves acquainted with basic notions around valued fields (e.g. Section 1–3.1 of [7]) beforehand which will be reviewed during the first two talks.

Talks.

(1) **Overview and introduction to valued fields** *(1-2 talks)*

These introductory talks give an overview of the basic algebraic theory of valued fields, following Sections 1–3.1 of [7]. They should cover valuations, valuation rings, residue fields, value groups, examples, extensions of valued fields, and the basic structure of henselian fields.

The talks are intended as a condensed summary of the background material. Participants are strongly encouraged to read [7] beforehand. *These talks may be given by the instructors.*

(2) **Ordered abelian groups** *(1 talk)*

Recall the model theory of divisible ordered abelian groups and Presburger arithmetic, including the corresponding quantifier elimination results; see, for example, Section 3 of [6]. Briefly explain the relationship between convex subgroups of the value group and coarsenings of valuations; see Section 2.3 of [10] or [4]. Finally, introduce NIP and state the theorem of Gurevich and Schmitt that ordered abelian groups are NIP [5]. If time permits, give a sketch of the proof.

(3) **Algebraically closed valued fields I: quantifier elimination** *(1 talk)*

Introduce algebraically closed valued fields and present Robinson's quantifier elimination theorem for ACVF, following Section 4 of [6]. The statement and proof of Theorem 4.4 should be the central part of the talk.

(4) **Algebraically closed valued fields II: definable sets and NIP** *(1 talk)*

Continue the study of ACVF by discussing the description of definable sets, especially Corollary 4.11 of [6]. If time permits, also explain why ACVF is NIP, following Theorem 4.14 of [6].

(5) **Henselian fields and Kaplansky theory** *(1-2 talks)*

The goal of this talk is to present the algebraic ingredients needed for Pas' quantifier elimination in the following talk. The choice of results should be guided by this goal. In particular, the talk should cover Theorem 5.6, without proof if necessary, Theorem 5.14, and Theorem 5.24 of [10]. Alternatively, the speaker may follow Section 5 of [6].

(6) **The Ax–Kochen–Ershov theorem and Pas' quantifier elimination** *(1 talk)*

Present the beginning of Section 6 of [6], up to and including Pas' quantifier elimination theorem.

(7) **Applications of Ax–Kochen–Ershov** *(1 talk)*

Start with Corollary 6.11 of [6]. Then present Theorem 6.17 of [6]. Finally, give an application to Artin's conjecture using the Ax–Kochen–Ershov theorem, following Section 6.3 of [10].

(8) **NIP transfer for valued fields** *(1 talk)*

The main result to present is Delon's theorem: a henselian valued field of residue characteristic zero is NIP if and only if its residue field is NIP. See Appendix A.15 and the subsequent proof in [12]. If time permits, the speaker may also briefly discuss the main result of [8].

(9) **Some model theory of $\mathbb{Q}_p$** *(1 talk)*

Present quantifier elimination for $\mathbb{Q}_p$ in Macintyre's language, including the proof. One possible source is Section 7.1 of [10]; the original paper is [9].

(10) **Further model theory of $\mathbb{Q}_p$** *(optional, 1 talk)*

Show that the theory of $\mathbb{Q}_p$ has definable Skolem functions, following Theorem 7.23 of [10]. Alternatively, one may follow Section 5.5 of [3]. If time permits, also discuss Section 5.6 of [3]. As an optional extension, the talk may include a brief introduction to p-adic cell decomposition.

(11) **Metric valued fields in continuous logic** *(optional, 1 talk)*

Give a brief introduction to the syntax of continuous logic, following Section 3 of [2]. Then present Ben Yaacov's formalization of metric valued fields in continuous logic, covering [1, Section 1.1–Theorem 1.8].

(12) **Algebraically closed metric valued fields** *(optional, 1 talk)*

State compactness for continuous logic, following [2, Corollary 5.12], and introduce stability in continuous logic, following [2, Definition 14.1]. Then present quantifier elimination for algebraically closed metric valued fields [1, Theorem 2.4] and, if feasible, the stability result [1, Theorem 2.10].

References

[1] Itaï Ben Yaacov. *Model theoretic properties of metric valued fields.* Journal of Symbolic Logic, 79(3):655–675, 2014. doi: https://doi.org/10.1017/jsl.2014.16. arXiv: https://arxiv.org/abs/0907.4560.

[2] Itaï Ben Yaacov, Alexander Berenstein, C. Ward Henson, and Alexander Usvyatsov. *Model theory for metric structures.* In Zoé Chatzidakis, Dugald Macpherson, Anand Pillay, and Alex Wilkie, editors, *Model Theory with Applications to Algebra and Analysis, Vol. 2*, London Mathematical Society Lecture Note Series, vol. 350, pages 315–427. Cambridge University Press, Cambridge, 2008. doi: https://doi.org/10.1017/CBO9780511735219.011.

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[5] Yuri Gurevich and Peter H. Schmitt. *The theory of ordered abelian groups does not have the independence property.* Transactions of the American Mathematical Society, 284(1):171–182, 1984. doi: https://doi.org/10.1090/S0002-9947-1984-0742419-0.

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[8] Itay Kaplan, Thomas Scanlon, and Frank O. Wagner. *Artin–Schreier extensions in NIP and simple fields.* Israel Journal of Mathematics, 185(1):141–153, 2011. doi: https://doi.org/10.1007/s11856-011-0104-7. Available at https://math.berkeley.edu/~scanlon/papers/NIPfields.pdf.

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[10] David Marker. *Model Theory of Valued Fields.* Lecture notes, University of Illinois at Chicago, Fall 2018. Available at https://homepages.math.uic.edu/~marker/valued_fields.pdf.

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